Interpolation Routines for Curves

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February 1, 2021

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Agenda

1. Research Background
2. Desirable features of any interpolation techniques
3. Quite comprehensive view of interpolation techniques
4. Different interpolation methodologies on zero, forward rates and discounting factors
5. Can linear methods have continuity in forward rates?
6. Spline methods and mentioning of Hermite interpolation as well as Hagan's monotone convex method

Research Background

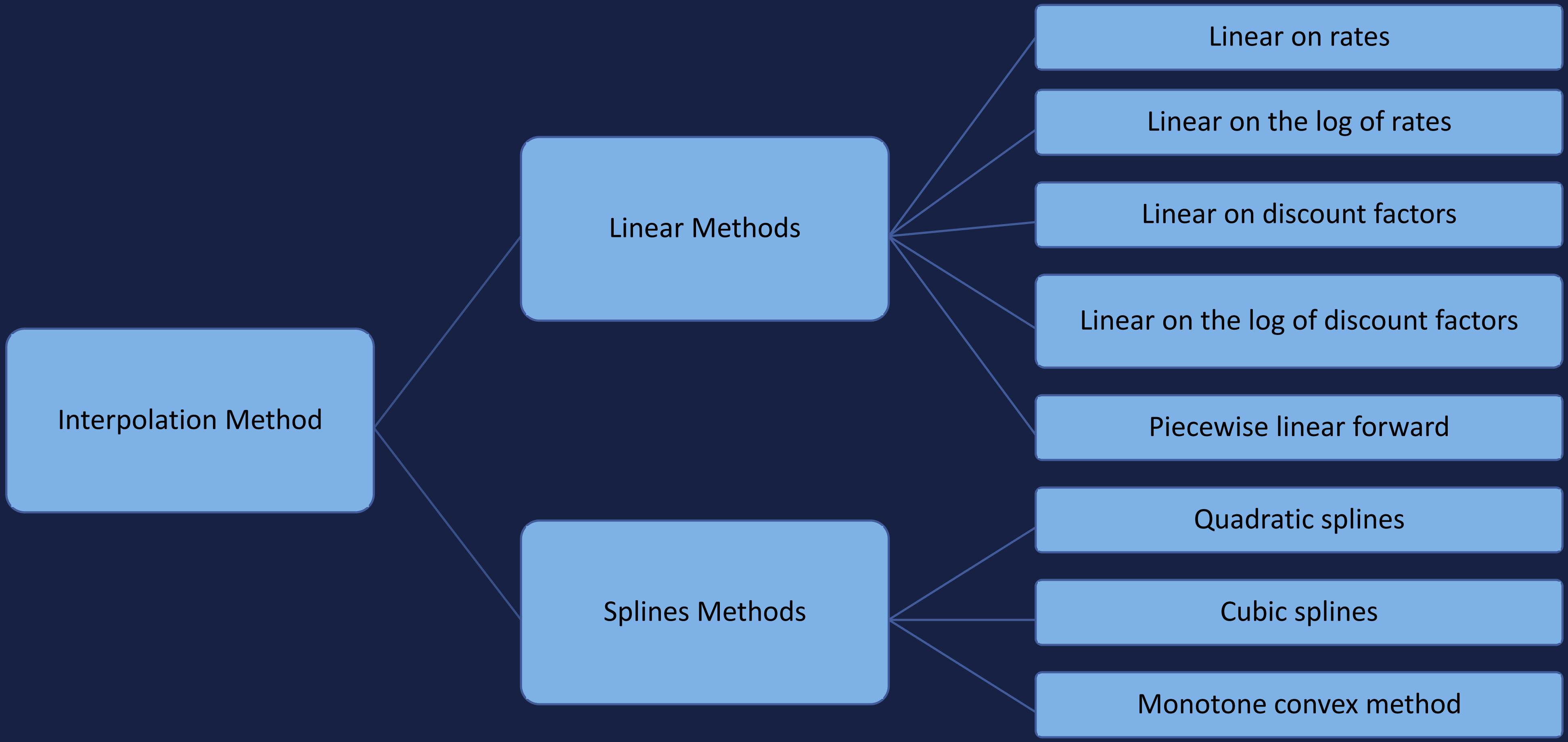
- The aim of this study is to compare different curve interpolation method. To decide performance of the different interpolation method, this study adopted the Hagan criteria as the measurement, which is proposed by Hagan and West in their paper published in 2016 in the WILMOTT magazine.
- This study will illustrate interpolation method among
  - linear methods
  - spline method
- based on three Hagan criteria:
  - the shape of the forward curve
  - the locality of the interpolation
  - the stability of the forward curve

How to Compare Curve Interpolation Method

Hagan Criteria

- In a paper published in 2006 in the WILMOTT magazine Hagan and West proposed a set of criteria to compare the interpolations method:
- The shape of the forward curve
  - Continuity of forward rate is preferred. Forwards should also usually be positive but in today's environment negative ones are possible (futures price > 100). While continuity is desirable but should not be achieved at the expense of locality and stability criteria.
- The locality of the interpolation
  - If an input of an instrument is changed, does the interpolation function only change nearby, with no or minor spill-over elsewhere, or can the changes elsewhere be material?
  - How local are hedging?
- The stability of the forward curve
  - The degree of stability can be quantified by looking for the maximum basis point change in the forward curve given some basis point change (up or down) in one of the inputs of instruments.

Interpolation Methodology



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Linear Method

- **Linear on Rate Interpolation**

For $t_{i-1} < t < t_i$, the interpolation formula:

$$r(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}} r_i + \frac{t_i - t}{t_i - t_{i-1}} r_{i-1}$$

The forward rate at time t would be:

$$f(t) = \frac{2t - t_{i-1}}{t_i - t_{i-1}} r_i + \frac{t_i \; - 2t}{t_i \; - \; t_i} r_{i-1}$$

- Zero Rate function r(t) is not differentiable.
- Interpolation does not prevent negative forward rate.
- Forward rate would jump at each point as left limit $f(t^-)$ and right limit $f(t^+)$ are different.

Linear Method

Linear on the Log of Rate Interpolation

For $t_{i-1} < t < t_i$, the interpolation formula:

$$\ln\left(r(t)\right) = \frac{t - t_{i-1}}{t_i - t_{i-1}} \ln(r_i) + \frac{t_i - t}{t_i - t_{i-1}} \ln(r_{i-1})$$

The rate at time t would be:

$$r(t) = r_i^{\frac{t - t_{i-1}}{t_i - t_{i-1}}} r_{i-1}^{\frac{t_i - t}{t_i - t_i}}$$

- Negative interest rate is not allowed (showstopper).
- Positive forward rate is not guaranteed.
- Forward rate would also jump at each point. Continuity on forward curve is poor.

Linear Method

- **Linear on Discount Factors Interpolation**

For $t_{i-1} < t < t_i$, the interpolation formula:

$$Z(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}} Z_i + \frac{t_i - t}{t_i - t_{i-1}} Z_{i-1}$$

The rate at time t would be:

$$r(t) = \frac{-1}{t} \ln\left[\frac{t - t_{i-1}}{t_i - t_{i-1}} e^{-r_i t_i} + \frac{t_i - t}{t_i - t_{i-1}} r^{-r_{i-1} t_{i-1}}\right]$$

- Forward rate would also jump at each point.
- Continuity on forward curve is poor.
- Discount Factor function may not be decreasing.
- So positive forward rate is also not guaranteed.

Linear Method

- **Linear on the log of discount factors Interpolation**

Raw interpolation is the method which has constant instantaneous forward rates:

$$f(t) = \frac{r_i t_i - r_{i-1} t_{i-1}}{t_i - t_{i-1}}$$

on every interval $t_{i-1} < t < t_i$.

For $t_{i-1} < t < t_i$, the interpolation formula:

$$r(t)t = r_{i-1} t_{i-1} + \int_{t_{i-1}}^{t} f(s)ds \;, t \in [t_{i-1}, t_i]$$

The rate at time t would be:

$$r(t)t = \frac{t - t_{i-1}}{t_i - t_{i-1}} r_i t_i + \frac{t_i - t}{t_i - t_{i-1}} r_{i-1} t_{i-1}$$

- The method is linear interpolation on the points $r_i\, t_i$.
- All instantaneous forwards are positive and has constant value on each interpolation interval.
- Only at the points $t_1 , t_2 , . . . , t_n$ that the instantaneous forward is undefined, and the function jumps at that point.
- This method is very stable, is trivial to implement, and is usually the starting point for developing models of the yield curve.

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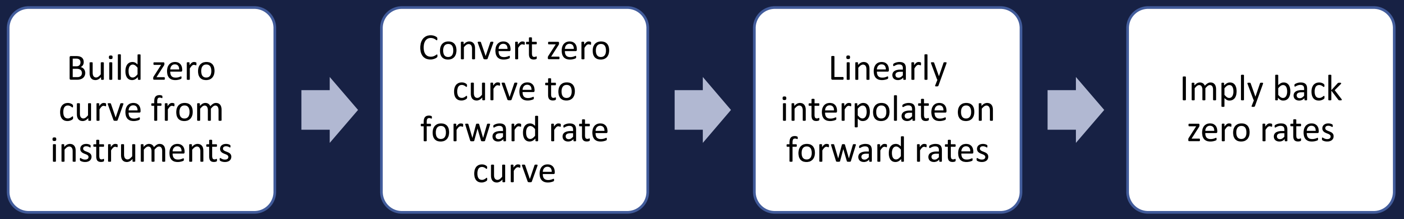
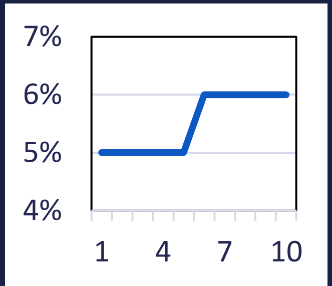
Linear Method and Continuity of Forwards – 1

- To recap, most illustrated methods have an issue with continuity in forwards. So, let’s try and just interpolate Piecewise Linear Forward and see what happens
- Achieving piecewise linear and continuity at same time. Is that possible?
- Let’s look at two examples and observe forward shape and locality
- Recall that our curves are always stored as zero rates
- Algorithm we apply is the following

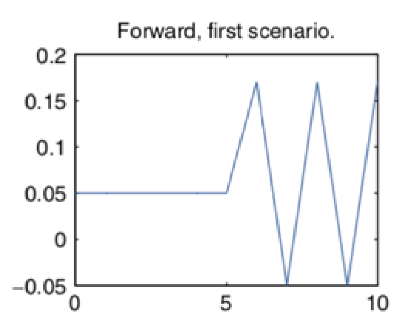


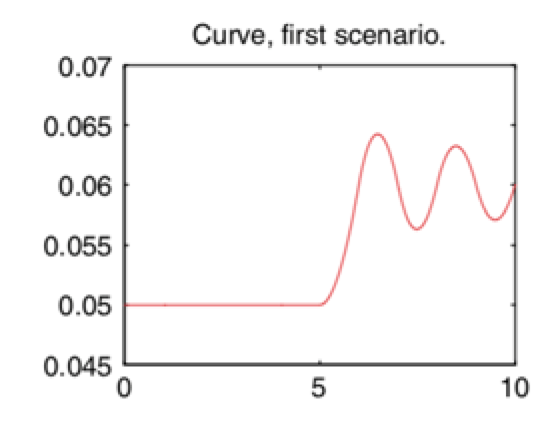
Linear Method and Continuity of Forwards – 2

Year	Zero Rate
1	5%
2	5%
3	5%
4	5%
5	5%
6	6%
7	6%
8	6%
9	6%
10	6%



Year	Fwd Rate
1	5%
2	5%
3	5%
4	5%
5	5%
6	17%
7	-5%
8	17%
9	-5%
10	17%

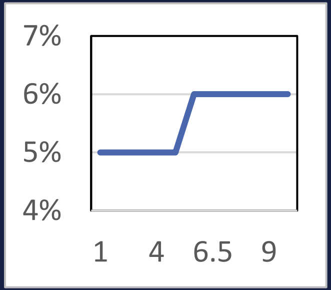




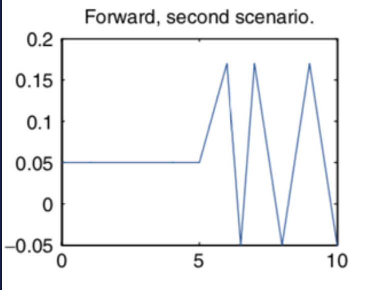
Linear Method and Continuity of Forwards – 3

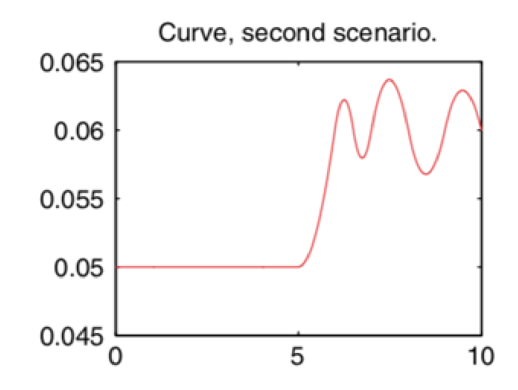
- Let's see what happens when we add a point without changing the curve's values
- We are adding 6.5 years point which keep the curve the same at 6% in this area

Year	Zero Rate
1	5%
2	5%
3	5%
4	5%
5	5%
6	6%
6.5	6%
7	6%
8	6%
9	6%
10	6%



Year	Fwd Rate
1	5%
2	5%
3	5%
4	5%
5	5%
6	17%
6.5	TBD
7	-5%
8	17%
9	-5%
10	17%





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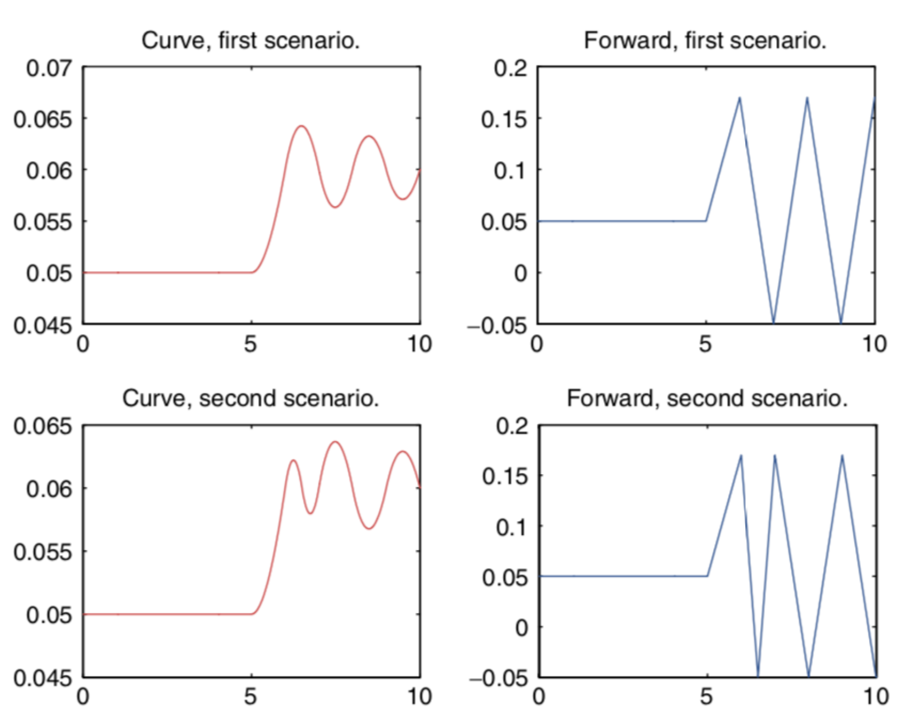
Linear Method and Continuity of Forwards – 4

- We asked if we could achieve piecewise linearity on forwards and continuity at same time by interpolating on forwards linearly. The answer is no as can be seen here
- Bad Forward Curve Shape. Can get a zig-zag fwds shape.
- Poor Localness. Changing 1 input can cause bootstrap curve to change dramatically.



Splines Methods

- **Quadratic Splines Method**

The complete a quadratic spline of a function x, we desire coefficients (a_i,b_i,c_i) for $1 \leq i \leq n-1$. Given these coefficients, the function value at any term τ will be

$$x(\tau) = a_i + b_i(\tau - \tau_i) + c_i(\tau - \tau_i)^2 \quad \tau_i \leq \tau \leq \tau_{i+1}$$

There are thus $3n - 4$ constraints:
$n - 1$ left hand function values to be satisfied,
$n - 1$ right hand function values to be satisfied,
$n - 2$ internal knots where differentiability needs to be satisfied.
With one degree of freedom, it makes sense to require that the left-hand derivative at τ_n be zero, so that the curve can be extrapolated with a horizontal asymptote.

- Entire function is continuous and differentiable.
- The forward curves that are produced are very similar to the Piecewise Linear Forward curves. The curve can have a 'zig-zag' appearance, and this zig-zag is subject to the same parity of input considerations as before.

Splines Methods

- **Cubic Splines Method**

This time we desire coefficients (a_i,b_i,c_i,d_i) for $1 \leq i \leq n-1$. Given these coefficients, the function value at any term τ will be

$$x(\tau) = a_i + b_i(\tau - \tau_i) + c_i(\tau - \tau_i)^2 + d_i(\tau - \tau_i)^3 \quad \tau_i \leq \tau \leq \tau_{i+1}$$

There are $4n - 4$ unknown coefficients and $3n - 4$ constraints:

- $n - 1$ left hand function values to be satisfied
- $n - 1$ right hand function values to be satisfied
- $n - 2$ internal knots where differentiability needs to be satisfied
- n more constrains would be needed to solve coefficient.

To find another n constraints , possible method are as follows：

- Natural Cubic Spline
- Financial Cubic Spline
- Quadratic-Natural Spline
- **Basel Method(Hermite Method)**
- Basel Interpolation
- Quartic Splines
- **Monotone Preserving Cubic Spline (Hyman[1983])**

Hermite and Monotone Convex

- The monotone convex method was developed to resolve the only remaining deficiency of Hyman.
- It explicitly ensures that the continuous forward rates are positive
- Its interpolation is directly performed on the forward curve instead of zero curve

$$f(\tau) = g\left(\frac{\tau-\tau_{i-1}}{\tau_i-\tau_{i-1}}\right) + f_i^d \cdot$$

- Where g is piecewise quadratic function

- Forward is positive and continuous
- Hermite is another interpolation technique on forward rates but goes beyond this course
- You should review the two methods and come back with questions

Conclusion

- A synopsis of the comparison between curve interpolation methods.

Yield curve type	Forwards positive?	Forward smoothness	Method local?	Forwards stable?	Bump hedges local?
Linear on discount	no	not continuous	excellent	excellent	very good
Linear on rates	no	not continuous	excellent	excellent	very good
Linear on log of discount	yes	not continuous	excellent	excellent	very good
Linear on the log of rates	no	not continuous	excellent	excellent	very good
Piecewise linear forward	no	continuous	poor	very poor	very poor
Quadratic	no	continuous	poor	very poor	very poor
Natural cubic	no	smooth	poor	good	poor
Hermite/Bessel	no	smooth	very good	good	poor
Financial	no	smooth	poor	good	poor
Quadratic natural	no	smooth	poor	good	poor
Hermite/Bessel on rt function	no	smooth	very good	good	poor
Monotone piecewise cubic	no	continuous	very good	good	good
Quartic	no	smooth	poor	very poor	very poor
Monotone convex (unameliorated)	yes	continuous	very good	good	good
Monotone convex (ameliorated)	yes	continuous	good	good	good
Minimal	no	continuous	poor	good	very poor

Desirable features of any interpolation techniques

Quite comprehensive view of interpolation techniques

Different interpolation methodologies on zero, forward rates and discounting factors

Can linear methods have continuity in forward rates?

Spline methods and mentioning of Hermite interpolation as well as Hagan’s monotone convex method